Lösungen:

A1 $f(x,y,z) = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$ gegeben

$$\frac{\partial f}{\partial x} \underset{KR}{=} \frac{2x}{2\cdot(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

übernehmen

analog $\frac{\partial f}{\partial y}$ und $\frac{\partial f}{\partial z}$ (aus Symmetriegründen)

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \underset{QR}{=} \frac{1\cdot\sqrt{x^2+y^2+z^2}^{\,*} - \frac{x}{\sqrt{x^2+y^2+z^2}}\cdot x}{\left(\sqrt{x^2+y^2+z^2}\right)^2}$$

* erweitert: $\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}$

$$= \frac{x^2+y^2+z^2 - x^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

analog (Symmetrie): $f_{yy} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ und $f_{zz} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$

zusammen: $f_{xx}+f_{yy}+f_{zz} = \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$

kürzen

$$= \frac{2}{f(x,y,z)}$$

A2 Es ist $F(x,y) = (x-2)^2+(y-1)^2-25 = 0$

$\hookrightarrow$ $F_x = 2\cdot(x-2)$

$F_y = 2\cdot(y-1)$

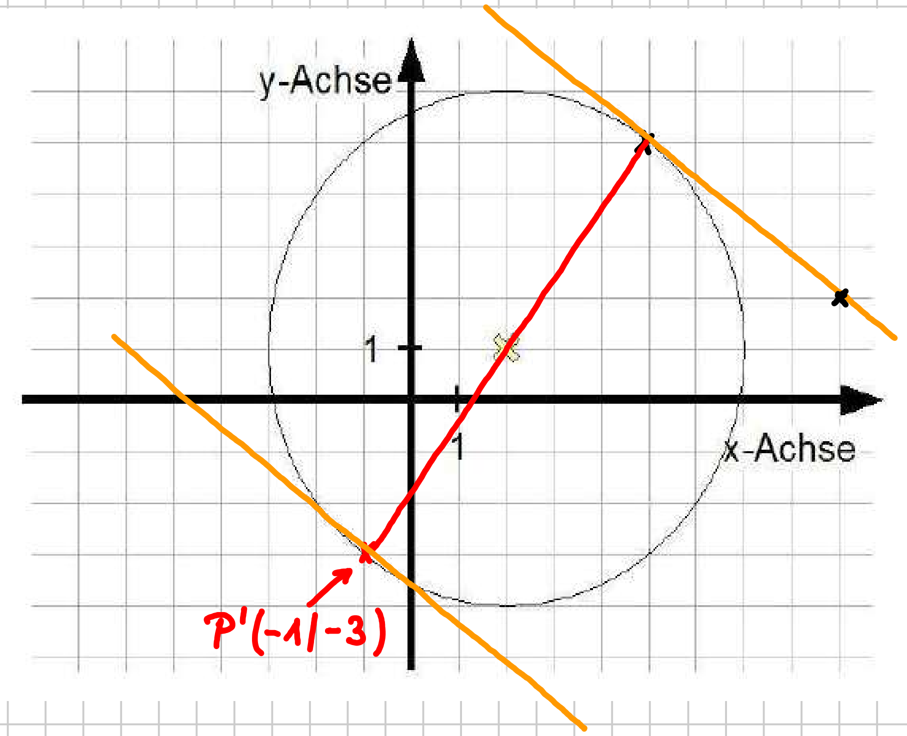
Steigung ist dann $y' = \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{x-2}{y-1}$

$x=5$: $(5-2)^2+(y-1)^2-25=0 \Rightarrow (y-1)^2=16 \quad |\sqrt{\ }, \ y>0$

$y-1=4 \Rightarrow y=5$

Damit ist $y' = -\frac{5-2}{5-1} = -\frac{3}{4}$

Skizze:



<u>A3</u> Tangentialebene: (Formel) $f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0) + z_0 = z$

mit $z = \cos((x^2+y^3)\cdot\pi) - \ln y = f(x,y)$

$\Rightarrow z_0 = \cos(\underbrace{(4+1)\cdot\pi}_{5\pi}) - \underbrace{\ln 1}_{=0} = \cos(5\pi) = -1$

$\uparrow$ $x=2, y=1$

$\Rightarrow P(2|1|-1)$

außen

$f_x = -\sin((x^2+y^3)\cdot\pi)\cdot 2x\pi \Rightarrow f_x(2,1) = \cancel{-\sin(5\pi)}^{\,0}\cdot 4\pi = 0$

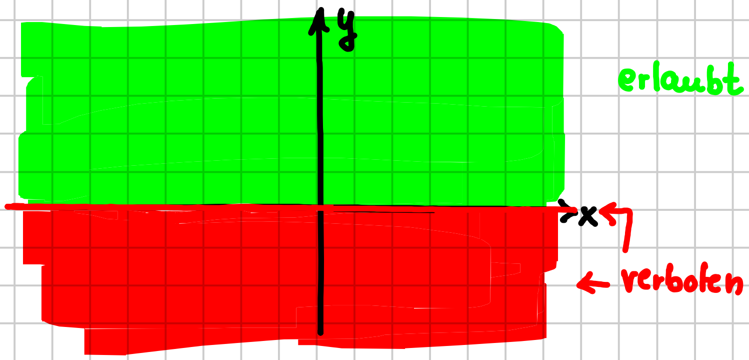
innen

siehe hier

$f_y = -\sin((x^2+y^3)\cdot\pi)\cdot 3y^2\pi - \frac{1}{y} \Rightarrow f_y(2,1) = 0 - \frac{1}{1} = -1$

Zusammen: $0\cdot(x-2) + (-1)\cdot(y-1) - 1 = z \Rightarrow y+z=0$

KEINE Tangentialebene: $\ln y \Rightarrow y > 0$



A4 Hilfsfunktion

$$h(x,y,z,\lambda) = x^2 + 4y^2 + 25z^2 + \lambda \cdot \underbrace{(x^2 + y^2 + z^2 - 100)}_{\varphi(x,y,z)}$$

$$\begin{aligned}
&\text{I} \quad h_x = 2x + 2\lambda x \overset{!}{=} 0 \\
&\text{II} \quad h_y = 8y + 2\lambda y \overset{!}{=} 0 \\
&\text{III} \quad h_z = 50z + 2\lambda z \overset{!}{=} 0 \\
&\text{IV} \quad h_\lambda = x^2 + y^2 + z^2 - 100 \overset{!}{=} 0
\end{aligned}$$

$! =$ Forderung

I - III:

$$\begin{aligned}
2x(1+\lambda) &= 0 \Rightarrow x = 0 \lor \lambda = -1 \\
2y(4+\lambda) &= 0 \Rightarrow y = 0 \lor \lambda = -4 \\
2z(25+\lambda) &= 0 \Rightarrow z = 0 \lor \lambda = -25
\end{aligned}$$

$$\left.\begin{aligned}
\lambda = -1: &\Rightarrow z = y = 0 \\
\lambda = -4: &\Rightarrow x = z = 0 \\
\lambda = -25: &\Rightarrow x = y = 0
\end{aligned}\right\} \text{mit } h_\lambda \text{ folgt: } \begin{aligned}
&P_{1/2}(\pm 10|0|0) \\
&P_{3/4}(0|\pm 10|0) \\
&P_{5/6}(0|0|\pm 10)
\end{aligned}$$

A5 $f(x,y) = x^4 - 2x^2 + (2x^2 - 1) \cdot y^2$

$$\begin{aligned}
&\text{I} \quad f_x = 4x^3 - 4x + 4xy^2 \overset{!}{=} 0 \\
&\text{II} \quad f_y = 2y \cdot (2x^2 - 1) \overset{!}{=} 0
\end{aligned}$$

Aus II folgt: $y=0$ und x beliebig **A**
$x=\pm\frac{\sqrt{2}}{2}$ und y beliebig **B**

Aus I folgt: $4x\cdot(x^2-1+y^2)=0$

mit **A**: $4x\cdot(x^2-1)=0 \Rightarrow x=0$ oder $x=\pm 1$

$\hookrightarrow$ $P_1(0|0)$; $P_2(-1|0)$; $P_3(+1|0)$

mit **B**: $4\cdot\frac{\sqrt{2}}{2}\cdot\left(\left(\frac{\sqrt{2}}{2}\right)^2-1+y^2\right)=0$

$\oplus$ $\frac{1}{2}-1+y^2=0$

$\Rightarrow y^2=\frac{1}{2}$, also $y_{1/2}=\pm\frac{\sqrt{2}}{2}$

$\hookrightarrow$ $P_4\left(\frac{\sqrt{2}}{2}\Big|\frac{\sqrt{2}}{2}\right)$; $P_5\left(\frac{\sqrt{2}}{2}\Big|-\frac{\sqrt{2}}{2}\right)$

$\ominus$ $-4\cdot\left(\frac{\sqrt{2}}{2}\right)\cdot\left(\left(-\frac{\sqrt{2}}{2}\right)^2-1+y^2\right)=0$

$\Rightarrow y^2=\frac{1}{2}$, also $y_{1/2}=\pm\frac{\sqrt{2}}{2}$

$\hookrightarrow$ $P_6\left(-\frac{\sqrt{2}}{2}\Big|\frac{\sqrt{2}}{2}\right)$; $P_7\left(-\frac{\sqrt{2}}{2}\Big|-\frac{\sqrt{2}}{2}\right)$

<u>2. Ableitungen:</u>

$$\left.\begin{aligned} f_{xx}&=12x^2-4+4y^2\\ f_{xy}&=f_{yx}=8xy\\ f_{yy}&=4x^2-2\end{aligned}\right\}\Rightarrow\begin{pmatrix}12x^2-4+4y^2 & 8xy\\ 8xy & 4x^2-2\end{pmatrix}$$

Einsetzen:

$P_1(0|0)$: $\begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 8 > 0$ und $-4 < 0 \Rightarrow$ Maximum

$P_2(-1|0)$: $\begin{vmatrix} 8 & 0 \\ 0 & 2 \end{vmatrix} = 16 > 0$ und $8 > 0 \Rightarrow$ Minimum

$P_3(1|0)$: $\begin{vmatrix} 8 & 0 \\ 0 & 2 \end{vmatrix} = 16 > 0$ und $8 > 0 \Rightarrow$ Minimum

$P_4\left(\frac{\sqrt{2}}{2} \middle| \frac{\sqrt{2}}{2}\right)$: $\begin{vmatrix} 4 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0 \Rightarrow$ Sattelpunkt

$P_5\left(\frac{\sqrt{2}}{2} \middle| -\frac{\sqrt{2}}{2}\right)$: $\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \Rightarrow$ Sattelpunkt

$P_6\left(-\frac{\sqrt{2}}{2} \middle| \frac{\sqrt{2}}{2}\right)$. $\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \Rightarrow$ Sattelpunkt

$P_7\left(-\frac{\sqrt{2}}{2} \middle| -\frac{\sqrt{2}}{2}\right)$: $\begin{vmatrix} 4 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0 \Rightarrow$ Sattelpunkt

A6

a) Gerade ist $y = mx = \frac{b}{a} \cdot x$ (Ursprungsgerade)

$$x_s = \frac{1}{A} \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}x} x \, dy\, dx = \frac{1}{A} \int_{x=0}^{a} [xy]_0^{\frac{b}{a}x} \, dx$$

$$= \frac{1}{A} \int_{x=0}^{a} \frac{b}{a} x^2 \, dx = \frac{1}{A} \cdot \frac{b}{a} \int_{x=0}^{a} x^2 dx = \frac{1}{A} \cdot \frac{b}{a} \cdot \left[\frac{1}{3} x^3\right]_0^a$$

$A = \frac{1}{2}ab \nearrow$ $$= \frac{2}{\cancel{a}\cancel{b}} \cdot \frac{\cancel{b}}{\cancel{a}} \cdot \frac{1}{3} a^{\cancel{3}} = \underbrace{\frac{2}{3} a}.$$

$$y_s = \frac{1}{A}\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}x} y\,dy\,dx = \frac{1}{A}\cdot\int_{x=0}^{a}\left[\frac{1}{2}y^2\right]_0^{\frac{b}{a}x} dx$$

$$= \frac{\cancel{2}}{a\cancel{b}}\cdot\int_{x=0}^{a}\frac{1}{\cancel{2}}\cdot\frac{b^{\cancel{2}}}{a^2}\cdot x^2\,dx = \frac{b}{a^3}\left[\frac{1}{3}x^3\right]_0^a = \underbrace{\frac{b}{3}}.$$

$$\Rightarrow \boxed{S\left(\frac{2}{3}a \,/\, \frac{1}{3}b\right)}$$

6 b) $$\int_{x=-1}^{4}\int_{y=0}^{1}\int_{z=0}^{\pi} xy^2\cdot\cos(xz)\,dz\,dy\,dx$$

① nach z: $$xy^2\cos(xz) \xrightarrow{\int} \left[\cancel{x}y^2\cdot\sin(xz)\cdot\frac{1}{\cancel{x}}\right]_{z=0}^{\pi}$$

$$= \left[y^2\cdot\sin(\pi x) - \cancelto{0}{y^2\cdot\sin(0)}\right]$$

② nach y: $$y^2\cdot\sin(\pi x) \xrightarrow{\int} \left[\frac{1}{3}y^3\cdot\sin(\pi x)\right]_{y=0}^{1}$$

$$= \frac{1}{3}\sin(\pi x)$$

③ nach x: $$\frac{1}{3}\sin(\pi x) \xrightarrow{\int} \left[-\frac{1}{3\pi}\cos(\pi x)\right]_{x=-1}^{4}$$

$$= -\frac{1}{3\pi}\cdot\left[\cos(4\pi) - \cos(-\pi)\right]$$

$$= -\frac{1}{3\pi}\left[1 - (-1)\right] = \underbrace{-\frac{2}{3\pi}}.$$

A7

a) $\vec{\nabla} f \rightarrow f_x,\ f_y,\ f_z$

$$f_x = \frac{1}{1+x^2}\cdot 2x\cdot\frac{1}{y}$$

$$f_y = \cos z - \frac{\ln(1+x^2)}{y^2} \quad\Rightarrow\quad \vec{\nabla} f = \begin{pmatrix} \frac{2x}{y\cdot(1+x^2)} \\ \cos z - \frac{\ln(1+x^2)}{y^2} \\ -y\cdot\sin z \end{pmatrix}$$

$$f_z = -y\cdot\sin z$$

b) 1) $\ln\left(\frac{1}{r}\right) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right) = \ln\left[(x^2+y^2)^{-1/2}\right] = -\frac{1}{2}\cdot\ln(x^2+y^2)$

2) $\phi_x = -\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = -\frac{x}{x^2+y^2}$, analog: $\phi_y = -\frac{y}{x^2+y^2}$

↳ Symmetrie

$$\Rightarrow \phi_{xx} \underset{QR}{=} -\frac{1\cdot(x^2+y^2)-2x\cdot x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\phi_{yy} = -\frac{1\cdot(x^2+y^2)-2y\cdot y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Delta\phi = \phi_{xx}+\phi_{yy} = \frac{\cancel{x^2}-\cancel{y^2}+\cancel{y^2}-\cancel{x^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = \underline{0.}$$