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$$y'' + 2y' + py = e^{-x}$$

hom. DGL: $y'' + 2y' + py = 0$

$\Rightarrow$ char. P. $\lambda^2 + 2\lambda + p = 0$

$$\lambda_{1/2} = \frac{-2 \pm \sqrt{4-4p}}{2} = -1 \pm \sqrt{1-p}$$

$\boxed{p=0}$ $\lambda_1 = -1+1 = 0$, $\lambda_2 = -1-1 = -2$

F.S. $\{1, e^{-2x}\}$

$\boxed{p=1}$ $\lambda_1 = \lambda_2 = -1 \Rightarrow$ F.S. $\{e^{-x}, xe^{-x}\}$

$\boxed{p>1}$ $\lambda_1 = -1 + j\sqrt{p-1}$

$\lambda_2 = -1 - j\sqrt{p-1}$

$\Rightarrow$ F.S. $\{e^{-x}\sin\sqrt{p-1}\,x,\ e^{-x}\cos\sqrt{p-1}\,x\}$

$\boxed{p<1}$ $\lambda_1 = -1 + \sqrt{1-p}$, $\lambda_2 = -1 - \sqrt{1-p}$

$\Rightarrow$ F.S. $\{e^{-x} e^{\sqrt{1-p}\,x},\ e^{-x} e^{-\sqrt{1-p}\,x}\}$

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für $\boxed{p \geq 1}$ und $\boxed{p < 1}$ Normalansatz für inh. DGL:

$$\left.\begin{aligned} y_p &= A e^{-x} \\ y_p' &= -A e^{-x} \\ y_p'' &= A e^{-x} \end{aligned}\right\} (*)$$

(*) einsetzen in inh. DGL:

$$A e^{-x} - 2 A e^{-x} + p A e^{-x} = e^{-x}$$

$$\Rightarrow \quad -A + pA = 1$$

$$\Rightarrow \quad A(p-1) = 1 \Rightarrow \boxed{A = \frac{1}{p-1}}$$

$$\Rightarrow \quad \boxed{y_p = \frac{1}{p-1} e^{-x}}$$

für $p = 1$ Resonanzfall (doppelte ~~Resonanz~~ NS)

Ansatz: $y_p = Ax^2 e^{-x}$

$$(*)\quad \left\{\begin{aligned} y_p' &= 2Axe^{-x} - Ax^2 e^{-x} \\ &= e^{-x}(2Ax - Ax^2) \\ y_p'' &= -e^{-x}(2Ax - Ax^2) + e^{-x}(2A - 2Ax) \\ &= e^{-x}(Ax^2 - 2Ax + 2A - 2Ax) \\ &= e^{-x}(Ax^2 - 4Ax + 2A) \end{aligned}\right.$$

einsetzen in inh. DGL:

$$e^{-x}(Ax^2 - 4Ax + 2A) + 2e^{-x}(2Ax - Ax^2) + Ax^2 e^{-x} = e^{-x}$$

$$\Rightarrow \cancel{Ax^2} - \cancel{4Ax} + 2A + \cancel{4Ax} - \cancel{2Ax^2} + \cancel{Ax^2} = 1$$

$$\Rightarrow A = \frac{1}{2} \Rightarrow y_p = \frac{1}{2}x^2 e^{-x}$$

insgesamt:

$$p=1: \quad y = e^{-x}\left(c_1 + c_2 x + \frac{1}{2}x^2\right)$$

$$p>1: \quad y = e^{-x}\left(c_1 \cos\sqrt{p-1}\,x + c_2 \sin\sqrt{p-1}\,x\right) + \frac{1}{p-1}e^{-x}$$

$$p<1: \quad y = e^{-x}\left(c_1 e^{\sqrt{1-p}\,x} + c_2 e^{-\sqrt{1-p}\,x}\right) + \frac{1}{p-1}e^{-x}$$

S. 87, A 19 (Eliminationsverfahren)

a) (1) $\dot{x} + x - y = 0$ $\quad x(0) = 1$

(2) $\dot{y} + 2x - y = 0$ $\quad y(0) = 0$

$\underset{\text{"(2)-(1)"}}{\Longrightarrow}$ (3) $\dot{y} - \dot{x} + x = 0$

(1) $\Leftrightarrow$ (4) $y = \dot{x} + x \Rightarrow$ (5) $\dot{y} = \ddot{x} + \dot{x}$

(5) in (3): $\ddot{x} + \dot{x} - \dot{x} + x = 0 \Leftrightarrow \ddot{x} + x = 0$

$\Rightarrow$ char. P. $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1/2} = \pm j$

$\Rightarrow$ (6) $\boxed{x(t) = C_1 \sin t + C_2 \cos t}$

(6) in (4): $\boxed{y(t) = C_1 \cos t - C_2 \sin t + C_1 \sin t + C_2 \cos t}$

$$\Rightarrow \vec{x}(t) = C_1 \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix} + C_2 \begin{pmatrix} \cos t \\ -\sin t + \cos t \end{pmatrix}$$

$$\vec{x}(0) = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Rightarrow C_2 = 1, \quad C_1 = -1$

$$\Rightarrow \boxed{\vec{x}(t) = \begin{pmatrix} \cos t - \sin t \\ -2 \sin t \end{pmatrix}}$$

$$\Rightarrow \quad \vec{x}(t) = K_1 \vec{v}^{(1)} + K_2 \vec{v}^{(2)}$$

$$= K_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} + K_2 \begin{pmatrix} \sin t - \cos t \\ 2 \sin t \end{pmatrix}$$

$$\vec{x}(0) = K_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \quad K_1 = 0 \;, \; K_2 = -1$$

$$\Rightarrow \boxed{\vec{x}(t) = \begin{pmatrix} \cos t - \sin t \\ -2\sin t \end{pmatrix}} \quad (\text{Lsg. d. AWP})$$

Das System:

$$(*) \quad \boxed{\vec{x}(t) = K_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} + K_2 \begin{pmatrix} \sin t - \cos t \\ 2\sin t \end{pmatrix}}$$

ist äquivalent zu $(**)$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ -\sin t + \cos t \end{pmatrix}} \quad (\text{nach a)})$$

$$\Updownarrow$$

$$\vec{x}(t) = \begin{pmatrix} \cos t \, (K_1 + K_2) + \sin t \, (K_2 + K_1) \\ 2K_1 \cos t + 2K_2 \sin t \end{pmatrix}$$

$$\text{mit} \quad \left.\begin{matrix} K_1 + K_2 =: c_1 \\ K_2 - K_1 =: c_2 \end{matrix}\right\} \Rightarrow \quad \begin{aligned} K_2 &= \frac{c_1 + c_2}{2} \\ K_1 &= \tfrac{1}{2}(c_1 - c_2) \end{aligned}$$

$$\Rightarrow \quad \vec{x}(t) = \begin{pmatrix} c_1 \sin t + c_2 \cos t \\ (c_1 - c_2)\cos t + (c_1 + c_2)\sin t \end{pmatrix}$$

$$= c_1 \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix}$$

b) (mit Matrizen)

$$\left.\begin{aligned}\dot{x} &= -x+y\\ \dot{y} &= -2x+y\end{aligned}\right\} \Rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \underbrace{\begin{pmatrix}-1 & 1\\ -2 & 1\end{pmatrix}}_{=:A}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$|A-\lambda E| = 0 \Rightarrow \begin{vmatrix}-1-\lambda & 1\\ -2 & 1-\lambda\end{vmatrix} = 0 \Leftrightarrow \lambda^2+1=0 \Rightarrow \lambda_{1/2} = \pm j$$

Berechne EV zum EW $\lambda_1 = j$:

$$\begin{array}{cc|cl} -1-j & 1 & 0 & \mid\cdot(-2)\\ -2 & 1-j & 0 & \mid\cdot(1+j)\\ \hline -1-j & 1 & 0 & \\ 0 & 0 & 0 & \end{array}$$

$$\Rightarrow \quad v_2 = \alpha\,;\ \alpha\in\mathbb{R}$$

$$v_1 = \frac{-\alpha}{-1-j} = \frac{\alpha(1-j)}{(1+j)(1-j)} = \frac{\alpha}{2}(1-j)$$

$$\Rightarrow \vec{\tilde{v}}^{(1)} = \frac{\alpha}{2}\begin{pmatrix}1-j\\ 2\end{pmatrix} \overset{\alpha=2}{\Longrightarrow} \vec{\tilde{v}}^{(1)} = \begin{pmatrix}1-j\\ 2\end{pmatrix} \quad \text{(kompl. EV)}$$

$$\Rightarrow \vec{\tilde{x}} = e^{jt}\begin{pmatrix}1-j\\ 2\end{pmatrix} = (\cos t + j\sin t)\begin{pmatrix}1-j\\ 2\end{pmatrix} =$$

$$= \begin{pmatrix}\cos t - j\cos t + j\sin t + \sin t\\ 2\cos t + 2j\sin t\end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix}\cos t + \sin t\\ 2\cos t\end{pmatrix}}_{=:\vec{v}^{(1)}} + j\underbrace{\begin{pmatrix}\sin t - \cos t\\ 2\sin t\end{pmatrix}}_{=:\vec{v}^{(2)}}$$

S. 87, A 20

$$y''' - 2y'' + 3y' - y = 0$$

$$y =: z_1$$
$$y' = \boxed{z_1' =: z_2}$$
$$y'' = \boxed{z_2' =: z_3}$$
$$y''' = \boxed{z_3' = z_1 - 3z_2 + 2z_3}$$

(Normalform)

$$\Longleftrightarrow \begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 2 \end{pmatrix}}_{=: A} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\boxed{\lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0}$$ (char. P.)

$$|A - \lambda E| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$$

$$\underset{\text{Entw. 1. Zeile}}{=} -\lambda \begin{vmatrix} -\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Longleftrightarrow \boxed{\lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0}$$