A1 a) $yy' - e^{2x} = 0 \iff yy' = e^{2x}$

$$\Rightarrow \int y\,dy = \int e^{2x}dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}e^{2x} + c_1 \;\Big|\cdot 2$$

$$\Rightarrow y^2 = e^{2x} + \underbrace{2c_1}_{=:K}$$

Wegen $y^2 > 0 \Rightarrow e^{2x} + K > 0$

Falls $K > 0 \Rightarrow e^{2x} + K > 0 \quad \forall x \in \mathbb{R}$

Falls $K < 0 \Rightarrow e^{2x} > -K \iff x > \frac{1}{2}\ln(-K)$

$$\Rightarrow \boxed{y = \pm\sqrt{e^{2x} + K}}$$

b) $y'\tan x = 2\sqrt{y} \Rightarrow \int \frac{dy}{\sqrt{y}} = 2\int \frac{\cos x}{\sin x}dx$

$$\Rightarrow 2y^{1/2} = 2\ln|\sin x| + c_1 \;|:2 \Rightarrow y^{1/2} = \ln|\sin x| + \underbrace{\frac{c_1}{2}}_{=:K}$$

$$\Rightarrow \boxed{y = \ln|\sin x| + K}$$

d) $e^{yy'} - x = 0 \iff e^{yy'} = x \Rightarrow yy' = \ln x$

$$\Rightarrow \int y\,dy = \int \ln x\,dx \Rightarrow \frac{1}{2}y^2 = x(\ln x - 1) + c_1$$

$$\Rightarrow y^2 = 2x(\ln x - 1) + \underbrace{2c_1}_{=:K}$$

$$\Rightarrow \boxed{y = \pm\sqrt{2x(\ln x - 1) + K}}$$

A1 c) $$x^3 dy + (y+1)^2 dy = 0$$

$$\Longrightarrow \int (y+1)^2 dy = - \int x^3 dx$$

$$\Longrightarrow \frac{1}{3}(y+1)^3 = -\frac{1}{4}x^4 + c_1 \quad \Big| \cdot 3$$

$$\Longrightarrow (y+1)^3 = -\frac{3}{4}x^4 + \underbrace{3c_1}_{=:K}$$

$$\Longrightarrow y+1 = \sqrt[3]{K - \frac{3}{4}x^4}$$

$$\Longrightarrow \boxed{y = \sqrt[3]{K - \frac{3}{4}x^4} - 1}$$

A2 a) (1) $y = (x-c)^2 \Rightarrow$ (2) $y' = 2(x-c)$

$\Rightarrow x - c = \frac{y'}{2} \Rightarrow$ (3) $c = x - \frac{y'}{2}$

$\xRightarrow[(3)\text{ in }(1)]{} y = \left(\frac{y'}{2}\right)^2 \Leftrightarrow y = \frac{y'^2}{2^2} \Leftrightarrow \boxed{4y - y'^2 = 0}$

A2 b) (1) $y = c(1+\cos x) \Rightarrow$ (2) $y' = -c\sin x$

$\Rightarrow c = -\frac{y'}{\sin x}$ (3) $\xRightarrow[(3)\text{ in }(1)]{} y = -\frac{y'}{\sin x}(1+\cos x)$

$\Rightarrow \boxed{y\sin x + y'(1+\cos x) = 0}$

A2 c) (1) $y = \ln[c(x-1)] \Rightarrow y' = \frac{1}{c(x-1)} \cdot c = \frac{1}{x-1}$

$\Leftrightarrow \boxed{y'(x-1) = 1}$

A3 a) $x + 2y = c \Leftrightarrow 2y = c - x \Leftrightarrow y = \frac{c}{2} - \frac{x}{2}$

$\Rightarrow y' = -\frac{1}{2}$

$\xRightarrow[y' \to -\frac{1}{y'}]{} -\frac{1}{y'} = -\frac{1}{2} \Leftrightarrow y' = 2 \Rightarrow \boxed{y = 2x + K}$

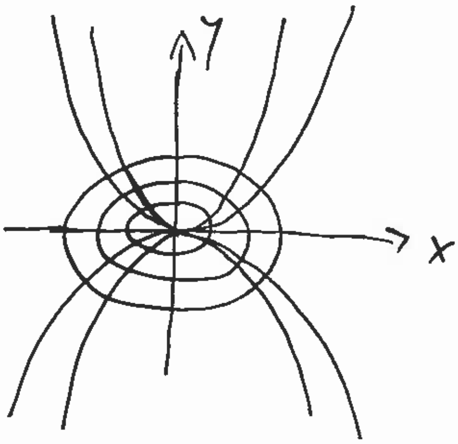
A3 b) $\quad (1)\quad x^2+2y^2=c \Longrightarrow 2x+4yy'=0 \;\big|:2$

$$\Longrightarrow x+2yy'=0 \xrightarrow[y' \to -\frac{1}{y'}]{} x-2y\cdot\frac{1}{y'}=0$$

$$\Longrightarrow 2y = xy' \Longrightarrow \int\frac{dy}{y} = 2\int\frac{dx}{x}$$

$$\Longrightarrow \ln|y| = 2\ln|x| + \ln K$$

$$\Longrightarrow \boxed{y = Kx^2}$$



A3 c) $\quad (1)\quad y = Ce^{-2x} \Longrightarrow (2)\quad y' = -2Ce^{-2x}$

$$\xrightarrow[y' \to -\frac{1}{y'}]{} -\frac{1}{y'} = -2Ce^{-2x} \Longrightarrow y' = \frac{e^{2x}}{2C}\ (3)$$

$$\xrightarrow[(1)]{} C = y\cdot e^{2x}\ (4) \xrightarrow[(4)\text{ in }(3)]{} y' = \frac{\cancel{e^{2x}}}{2y\cancel{e^{2x}}}$$

$$\Longrightarrow y'y = \frac{1}{2} \Longrightarrow \int y\,dy = \frac{1}{2}\int dx$$

$$\Longrightarrow \frac{1}{2}y^2 = \frac{1}{2}x + C_1 \Longrightarrow y^2 = x + \underbrace{2C_1}_{=:k}$$

$$\Longrightarrow \boxed{y = \pm\sqrt{x+K}}$$

A4

a) (1) $y' = (x+y)^2$ Typ: $y' = f(ax+by+c)$

(2) $z := x+y \Rightarrow$ (3) $y = z - x \Rightarrow y' = z' - 1$ (4)

$\underset{(4)\ \text{in}\ (1)}{\Longrightarrow} \quad z' - 1 = z^2 \Rightarrow z' = 1 + z^2 \Rightarrow \int \frac{dz}{1+z^2} = \int dx \Rightarrow$

$\arctan z = x + C \Rightarrow z = \tan(x+C) \underset{\text{R.S.}}{\Longrightarrow} x + y = \tan(x+C)$

$\Rightarrow \boxed{y = -x + \tan(x+C)}$

c) $2xyy' + x^2 - y^2 = 0 \mid : x^2 \ (\neq 0) \Rightarrow 2\frac{y}{x}y' + 1 - \left(\frac{y}{x}\right)^2 = 0$

$\Rightarrow$ (1) $y' = \dfrac{\left(\frac{y}{x}\right)^2 - 1}{2\frac{y}{x}}$ Typ: $y' = f\left(\frac{y}{x}\right)$

(2) $z := \frac{y}{x} \Rightarrow$ (3) $y = xz \Rightarrow$ (4) $y' = z + xz'$

$\underset{(2),(4)\ \text{in}\ (1)}{\Longrightarrow} \quad z + xz' = \frac{z^2-1}{2z} \Rightarrow xz' = \frac{z^2-1}{2z} - z = \frac{z^2-1-2z^2}{2z}$

$\Rightarrow xz' = -\frac{z^2-1}{2z} = -\frac{(z^2+1)}{2z}$

$\Rightarrow \int \frac{2z\,dz}{1+z^2} = \int \frac{dx}{x} \Rightarrow \ln(1+z^2) = \ln|x| + \ln C \quad (C>0)$

$\Rightarrow \ln(1+z^2) = \ln\frac{C}{|x|} \Rightarrow 1+z^2 = \frac{\pm C}{x}$, $\pm C =: C_1$ $\quad C_1 \in \mathbb{R}$

$\underset{\text{R.S.}}{\Longrightarrow} 1 + \frac{y^2}{x^2} = \frac{C_1}{x} \Leftrightarrow x^2 + y^2 = C_1 x$ $\quad$ Kreis um $\left(\frac{C_1}{2} \middle| 0\right)$

$\Leftrightarrow x^2 - C_1 x + \left(\frac{C_1}{2}\right)^2 + y^2 = \left(\frac{C_1}{2}\right)^2 \Leftrightarrow \left(x - \frac{C_1}{2}\right)^2 + y^2 = \left(\frac{C_1}{2}\right)^2$ $\quad r = \frac{C_1}{2}$

A4 b)

$$(2x-y+3)y' = 1 \iff y' = \frac{1}{2x-y+3} \quad (1)$$

Typ $\quad y' = f(ax+by+c) \quad (2) \quad z := 2x-y+3$

$$\Rightarrow \quad y = 2x+3-z \Rightarrow (3) \quad y' = 2-z'$$

$$\underset{\substack{(2),(3)\\ \text{in } (1)}}{\Longrightarrow} \quad 2-z' = \frac{1}{z} \Rightarrow z' = 2-\frac{1}{z} = \frac{2z-1}{z}$$

$$\Rightarrow \int \frac{z}{2z-1}\,dz = \int dx \Rightarrow \frac{1}{2}\int \frac{2z-1+1}{2z-1}\,dz = \int dx$$

$$\Rightarrow \frac{1}{2}\int\left(1+\frac{1}{2z-1}\right)dx = \int dx \Rightarrow \frac{1}{2}z + \frac{1}{2}\cdot\frac{1}{2}\ln|2z-1| = x + C_1$$

$$\iff 2z + \ln|2z-1| = 4x + 4C_1 =: C_2$$

$$\underset{\text{R.S.}}{\Longrightarrow} \quad \cancel{4x} - 2y + 6 = \ln|4x-2y+5| = \cancel{4x} + C_2$$

$$\iff \ln|4x-2y+5| = -2y + C_3 \quad \text{mit} \quad C_3 := 6 - C_2$$

$$\Rightarrow |4x-2y+5| = e^{-2y}\cdot e^{C_3} \Rightarrow 4x-2y+5 = \underbrace{\pm e^{C_3}}_{=:C_4} e^{-2y}$$

$$\iff \boxed{e^{2y}(4x-2y+5) = C_4}$$

A 4 d) $\quad xy' + \sqrt{x^2+y^2} = y \quad \Big|\, : x \ (\neq 0)$

$\iff \quad y' + \sqrt{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x} \iff (1) \quad y' = -\sqrt{1+\left(\frac{y}{x}\right)^2} + \frac{y}{x}$

Typ $y' = f\left(\frac{y}{x}\right) \qquad (2) \quad z := \frac{y}{x} \implies (3) \quad y = x \cdot z$

$\implies (4) \quad y' = z + xz' \xRightarrow[(2),(3) \text{ in } (1)]{} \cancel{z} + xz' = -\sqrt{1+z^2} + \cancel{z}$

$\iff \quad xz' = -\sqrt{1+z^2} \implies \int \frac{dz}{\sqrt{1+z^2}} = -\int \frac{dx}{x}$

$\implies \ln\left|z+\sqrt{1+z^2}\right| = -\ln|x| + \ln C = \ln \frac{C}{|x|} \quad (C>0)$

$\implies \quad z+\sqrt{1+z^2} = \frac{C}{x} \implies \sqrt{1+z^2} = \frac{C}{x} - z \implies 1+z^2 = \frac{(C-zx)^2}{x^2}$

$\implies$ R.S. $\quad 1+\left(\frac{y}{x}\right)^2 = \frac{\left(C-\frac{y}{x}\cdot x\right)^2}{x^2} \implies \frac{x^2+y^2}{x^2} = \frac{(C-y)^2}{x^2} \quad \Big|\cdot x^2$

$\implies \quad x^2+y^2 = C - 2Cy + y^2$

$\implies \quad 2Cy = C^2 - x^2 \quad \Big|\, : 2C$

$\implies \quad \boxed{y = \frac{C^2-x^2}{2C} = \frac{1}{2}\left(C - \frac{x^2}{C}\right) \qquad (C>0)}$

5a) $y' + \frac{y}{1+x} + 6x = 0 \quad (*) \qquad ; \; y(0) = 3$

$$\Longleftrightarrow \underbrace{y' + \frac{y}{1+x}}_{=: L[y]} = -6x$$

1. Schritt: $L[y] = 0 \Rightarrow y' + \frac{y}{1+x} = 0$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{1+x} \Rightarrow \ln|y| = -\ln|1+x| + \ln c \quad (c>0)$$

$$\Rightarrow \boxed{y_h = \frac{c}{1+x}}$$

2. Schritt: $L[y] = -6x$ (Var. d. Konst.) $\quad y = \frac{C(x)}{1+x} \; (1)$

$$\Rightarrow y' = \frac{C'(1+x) - C}{(1+x)^2} \; (2)$$

$$\underset{(1),(2) \text{ in } (*)}{\Longrightarrow} \frac{C'(1+x) - C}{(1+x)^2} + \frac{C}{(1+x)^2} = -6x \Rightarrow \frac{C'}{1+x} = -6x$$

$$\Rightarrow C' = -(1+x)6x = -6x - 6x^2 \Rightarrow C = -3x^2 - 2x^3$$

$$\Rightarrow y_p = \frac{-3x^2 - 2x^3}{1+x} \Rightarrow y = y_h + y_p = \frac{c}{1+x} - \frac{3x^2 + 2x^3}{1+x}$$

$$y(0) = c \overset{!}{=} 3 \Rightarrow \boxed{y = \frac{3}{1+x} - \frac{3x^2 + 2x^3}{1+x} = \frac{3 - 3x^2 - 2x^2}{1+x}}$$

5c) (*) $\underbrace{y' + 2xy}_{=:L[y]} = 2x \quad ; \quad y(0) = 2$

1. Schritt: $L[y] = 0 \Rightarrow y' + 2xy = 0 \Rightarrow \int \frac{dy}{y} = -2 \int x\,dx$

$\Rightarrow \ln|y| = -x^2 + C_1 \Rightarrow y_h = Ce^{-x^2}$

2. Schritt $L[y] = r(x)$ (Var. d. Konst) $y = C(x)e^{-x^2}$ (1)

$\Rightarrow y' = C'e^{-x^2} + Ce^{-x^2}(-2x)$ (2)

$\xRightarrow[\text{in (*)}]{(1),(2)} C'e^{-x^2} - 2xCe^{-x^2} + 2xCe^{-x^2} = 2x \Rightarrow C'e^{-x^2} = 2x$

$\Longrightarrow C = \int 2xe^{+x^2}dx = e^{x^2}$

$\Longrightarrow y_p = 1$

$\Longrightarrow y = y_h + y_p = Ce^{-x^2} + 1 \; ; \; y(0) = C + 1 \overset{!}{=} 2 \Rightarrow C = 1$

$\Longrightarrow \boxed{y = e^{-x^2} + 1}$

5b) $y' \sin x = y \cos x \; ; \; y(\frac{\pi}{6}) = 1 \Longrightarrow \int_1^y \frac{ds}{s} = \int_{\frac{\pi}{6}}^x \frac{\cos t}{\sin t} dt$

$\Longrightarrow [\ln|s|]_1^y = [\ln(\sin t)]_{\frac{\pi}{6}}^x \Longrightarrow \ln|y| = \ln|\sin x| - \ln\frac{1}{2}$

$\Longrightarrow \ln|y| = \ln|\sin x| + \ln 2 = \ln|2\sin x|$

$\Longrightarrow \boxed{y = 2\sin x}$

A5 d) $y' \cdot x \cdot \ln x = y \qquad (x>0)$

$y(e^2) = 1$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x \ln x} = \int \frac{\frac{1}{x}}{\ln x}\, dx$$

$$\Rightarrow \ln|y| = \ln|\ln x| + \ln C \qquad (C>0)$$

$$\Rightarrow \ln|y| = \ln|C \cdot \ln x|$$

$$\Rightarrow \ln|y| = C \ln x$$

$$\Rightarrow y = \underbrace{\pm C}_{=:K} \ln x = K \ln x \qquad (K \in \mathbb{R})$$

$$y(e^2) = K \ln e^2 = 2K \underbrace{\ln e}_{=1} \overset{!}{=} 1 \Rightarrow K = \frac{1}{2}$$

$$\Rightarrow \boxed{y = \frac{1}{2} \ln x}$$

A6 a) $(*)\quad \underbrace{y' - \frac{y}{x}}_{=: L[y]} = x$

1. Schritt: $L[y] = 0 \Rightarrow y' - \frac{y}{x} = 0 \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$

$\Rightarrow \ln|y| = \ln|x| + \ln C \Rightarrow \boxed{y = Cx = y_h}$

2. Schritt: $L[y] = x$ (Var. d. Konst.) $y = C(x) \cdot x \quad (1)$

$\Rightarrow (2) \quad y' = C'x + C$

$\xRightarrow[\text{in } (*)]{(1), (2)} \quad C'x + C - C = x \Rightarrow C' = 1 \Rightarrow C = x$

$\Rightarrow y_p = x^2 \Rightarrow y = y_h + y_p = Cx + x^2$

A6 b) $y' = 0 \Rightarrow -\frac{y}{x} = x \Rightarrow \boxed{y = -x^2}$

A6 c) $y = x^2 + Cx = x^2 + Cx + \left(\frac{C}{2}\right)^2 - \left(\frac{C}{2}\right)^2$

$= \left(x + \frac{C}{2}\right)^2 - \left(\frac{C}{2}\right)^2$ Parabeln

mit Scheitel $S\left(-\frac{C}{2} \middle| \left(\frac{C}{2}\right)^2\right)$

d.h. $S\left(-\frac{C}{2} \middle| -\frac{C^2}{4}\right)$

Lsg.kurve durch $P(-2|0)$: $y(-2) = -2C + 4 = 0 \Rightarrow C = 2$

$\Rightarrow y = x^2 + 2x = x^2 + 2x + 1 - 1 = (x+1)^2 - 1$

A7a) $y'' \cos x + y' \sin x = 0 \qquad y(0) = 1 \qquad y'(0) = 2$

$$z := y' \Rightarrow z' = y'' \Longrightarrow z' \cos x + z \sin x = 0$$

$$\Rightarrow \int \frac{dz}{z} = -\int \frac{\sin x}{\cos x} dx \Rightarrow \ln|z| = \ln|\cos x| + \ln c_1$$

$$\Rightarrow z = K \cos x \underset{R.S.}{\Rightarrow} y' = K\cos x \Rightarrow y = K \int \cos x \Rightarrow y = K \sin x + M$$

$$\left.\begin{array}{l} \Rightarrow y' = K\cos x \Rightarrow y(0) = M \overset{!}{=} 1 \Rightarrow M = 1 \\ y'(0) = K \overset{!}{=} 2 \Rightarrow K = 2 \end{array}\right\} \Rightarrow \boxed{y = 2\sin x + 1}$$

A7b) (*) $2xy'' - y' = 9x^2 \quad (x \geq 0)$

$$(1)\ z := y' \Rightarrow (2)\ z' = y'' \underset{(1),(2) \text{ in } (*)}{\Longrightarrow} 2xz' - z = 9x^2$$

1. Schritt: hom DGL: $2xz' - z = 0 \Rightarrow \int \frac{dz}{z} = \frac{1}{2} \int \frac{dx}{x}$

$$\Rightarrow \ln|z| = \frac{1}{2} \ln x + \ln c_1 = \ln(\sqrt{x} \cdot c_1)$$

$$\Rightarrow z_h = C\sqrt{x} \quad (C \in \mathbb{R})$$

2. Schritt inh. DGL: Var. d. Konst.

$$z = \sqrt{x}\, C(x) \Rightarrow z' = \frac{1}{2\sqrt{x}} C(x) + \sqrt{x}\, C'(x)$$

$$\Rightarrow 2x \frac{1}{2\sqrt{x}} C(x) + 2x\sqrt{x}\, C'(x) - \sqrt{x}\, C(x) = 9x^2$$

$$\Rightarrow C'(x) \cdot 2x^{3/2} = 9x^2 \Rightarrow C'(x) = \frac{9}{2} x^{1/2}$$

$$\Rightarrow C(x) = \frac{9}{2} \int x^{1/2} dx = \frac{9}{2} \cdot \frac{2}{3} x^{3/2} = 3x^{3/2}$$

$$\Rightarrow z_p = 3x^{3/2}\sqrt{x} = 3x^2$$

$$\Rightarrow z = z_h + z_p = C\sqrt{x} + 3x^2$$

$$\underset{R.S.}{\Rightarrow} y' = C\sqrt{x} + 3x^2 \Rightarrow y = \int (C\sqrt{x} + 3x^2)\, dx$$

$$= \underbrace{C \cdot \frac{2}{3}}_{=: c_1} x^{3/2} + x^3 + c_2 = \boxed{x^3 + c_1 x^{3/2} + c_2 = y}$$