Bsp.: (freier Fall aus großer Höhe)

$g = g(h)$; $h = h(t)$ ... Höhe des Objektes zur Zeit t.

$\underset{\text{Gravitationsgesetz}}{\Longrightarrow}$ $\boxed{m\ddot{h} = -m \frac{MG}{h^2}}$

$g = g(h) = \frac{MG}{h^2}$; M ... Erdmasse ($\approx 5{,}97 \cdot 10^{27}\, g$)

G ... Grav. konst. ($\approx 6{,}67 \cdot 10^{-14} \frac{m^3}{g \cdot s^2}$)

DGL (*) $\boxed{\ddot{h} = -\frac{MG}{h^2}}$ $\boxed{h(0) = h_0,\ \dot{h}(0) = v_0}$ (AWP)

Lösungsversuch von (*): Mult. von (*) mit $\dot{h}$.

$$\Longrightarrow \ddot{h}\dot{h}\,dt = -\frac{MG}{h^2}\cdot\dot{h}\,dt = -\frac{MG}{h^2}\,dh$$

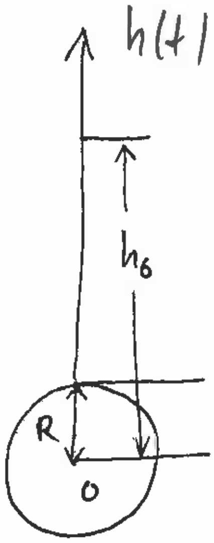
$$\underset{\int}{\Longrightarrow} \int \ddot{h}\dot{h}\,dt + \int \frac{MG}{h^2}\,dh = C_1$$

$$\Longrightarrow \underbrace{\frac{1}{2}(\dot{h}(t))^2}_{E_{kin}} - \underbrace{\frac{MG}{h}}_{E_{pot}} = C_1 \Longrightarrow (\dot{h}(t))^2 = \frac{2MG}{h} + C$$

$$\underset{\substack{h(0)=h_0\\ \dot{h}(0)=v_0}}{\Longrightarrow} (\dot{h}(t))^2 = v_0^2 + 2MG\left(\frac{1}{h(t)} - \frac{1}{h_0}\right)$$ (noch keine explizite DGL)

$$v := \dot{h}(t) = \pm\sqrt{v_0^2 + 2MG\left(\frac{1}{h(t)} - \frac{1}{h_0}\right)}$$

+ ... Köper steigt; – ... Körper fällt



Spezialfälle:

(a) "Ein Meteor fällt auf die Erde"

A.B. $h_0 = \infty$; $V_0 = 0$

Wir ermitteln die Aufprallgeschw.

$$h(t_A) \overset{!}{=} R \Rightarrow \boxed{\dot h(t_A) = \sqrt{\frac{2MG}{R}} \approx 11{,}2 \, \frac{km}{s}}$$

(b) "Fluchtgeschw"

Für $h_0 = R$ ist $\dot h(0)$ ist $\dot h(0) = V_0$ gesucht, so dass $h(t) \to \infty$ für $t \to \infty$

Notwendig: $v \geq 0 \Rightarrow v^2 = V_0^2 - \frac{2MG}{R} \geq 0$

$$\Rightarrow \boxed{V_0 \overset{!}{\geq} \sqrt{\frac{2MG}{R}}}$$

Als DGL: $\dot h = +\sqrt{\frac{2MG}{h}}$, $h(0) = R$, $\dot h(0) = V_0$

$$\Rightarrow \boxed{h(t) = \left(R^{3/2} + \frac{3}{2}\sqrt{2GM}\cdot t\right)^{2/3}}$$ (Ortsfunktion)

Bsp. (freier Fall mit Newton-Reibung)

$$(*) \quad \boxed{m\ddot{s} = mg - c\dot{s}^2} \qquad \boxed{s(0) = 0\,;\; \dot{s}(0) = 0} \qquad \text{(AWP)}$$

Typ $y'' = f(y')$, $(1)\; \dot{s} = z \Rightarrow \ddot{s} = \frac{dz}{dt}\; (2)$

$$\underset{(1),(2)\text{ in }(*)}{\Longrightarrow} \quad m\frac{dz}{dt} = mg - cz^2 \Longrightarrow m\int \frac{dz}{mg - cz^2} = \int dt$$

$$\Longrightarrow \cancel{m}\int \frac{dz}{\cancel{m}g\left(1 - \left(\sqrt{\frac{c}{mg}}\cdot z\right)^2\right)} = \int dt \qquad \left[\int \frac{dx}{1-x^2} = \operatorname{artanh} x + C_1\right]$$

$$\Longrightarrow \frac{1}{g}\,\frac{1}{\sqrt{\frac{c}{mg}}}\operatorname{artanh}\left(\sqrt{\frac{c}{mg}}\,z\right) = t + C_1$$

$$\Longrightarrow \sqrt{\frac{m}{cg}}\operatorname{artanh}\left(\sqrt{\frac{c}{mg}}\cdot z\right) = t + C_1\,; \quad \text{Weil } z = \dot{s}(0) = 0 \Rightarrow C_1 = 0$$

$$\Longrightarrow \operatorname{artanh}\left(\sqrt{\frac{c}{mg}}\cdot z\right) = \sqrt{\frac{cg}{m}}\cdot t \qquad \left[\int \tanh x\,dx = \int \frac{\operatorname{sh} x}{\operatorname{ch} x}\,dx + C_1\right]$$

$$\Longrightarrow \sqrt{\frac{c}{mg}}\,z = \tanh\left(\sqrt{\frac{cg}{m}}\,t\right)$$

$$\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{c}}$$

Grenzwert - Endgeschw.

$$\Longrightarrow p = \frac{ds}{dt} = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right) = v = v(t)$$

$$\Longrightarrow \boxed{\begin{aligned} s &= \sqrt{\frac{mg}{c}}\int \tanh\left(\sqrt{\frac{cg}{m}}\,t\right) + C_2 \\ &= \frac{m}{c}\ln\left(\cosh\left(\sqrt{\frac{cg}{m}}\,t\right)\right) + C_2 = s(t) \end{aligned}}$$

$C_2 = 0$, wegen $s(0) = 0$

Logistisches Wachstum (Bsp. zu BERNOULLI-DGL)

DGL $\boxed{\dot{B}(t) = k\,B(t)\,(S - B(t))}$, $k>0$; $S>0$; $B(0) = B_0$

t … Zeit $B(t)$ … Bestand zum Z.P. t, S Schranke

kurz: (1) $\dot{B} = kB(S-B) = kSB - kB^2$

(Bernoullische DGL)

(2) $B := \frac{1}{u} = u^{-1} \Rightarrow$ (3) $\dot{B} = -\frac{1}{u^2}\dot{u}$

(2), (3) in (1) $\Rightarrow -\frac{1}{u^2}\dot{u} = \frac{kS}{u} - k\cdot\frac{1}{u^2} \quad \Big|\cdot(-u^2)$

$\Rightarrow \dot{u} = -kSu + k = k(1-Su)$

$\Rightarrow \frac{\dot{u}}{1-Su} = k\,dt \Rightarrow \int \frac{du}{1-Su} = \int k\,dt + C_1$

$\Rightarrow \ln|1-Su| = kt + C_1$

$\left[\;\underbrace{1-\underbrace{\frac{S}{B}}_{>1}}_{<0}\;\right]$ $S>B$

$\Rightarrow \ln(Su-1) = kt + C_1 \quad \big|\exp$

$\overset{}{\underset{\exp}{\Longrightarrow}} \; Su - 1 = e^{kt}\cdot \underbrace{e^{C_1}}_{=:C>0}$

$\Rightarrow u = \frac{1}{S}\left(1 + Ce^{-kt}\right)$

$\Rightarrow B = \frac{S}{1+Ce^{-kt}} = B(t)$

$B(0) = \frac{S}{1+C} \overset{!}{=} B_0 \Leftrightarrow 1 + C = \frac{S}{B_0} \Leftrightarrow C = -1 + \frac{S}{B_0}$

$\Rightarrow \boxed{B(t) = \frac{SB_0}{B_0 + (S-B_0)e^{-kt}} = \frac{S}{1+\left(\frac{S}{B_0}-1\right)e^{-kt}}}$; $0 < B_0 < S$

Bsp.: (Kugel fällt in eine zähe Flüssigkeit)
(mit Stokes-Reibung)

DGL: $-m\frac{dv}{dt} - cv + mg = 0 \;\Big|\; :m \quad (m \neq 0)$

$$\Longleftrightarrow (*) \boxed{\frac{dv}{dt} + \frac{c}{m}v = g}, \quad \boxed{v(0) = v_0} \quad \text{(AWP)}$$

1. Schritt: (hom. DGL) $\frac{dv}{dt} + \frac{c}{m}v = 0 \Longleftrightarrow \frac{dv}{v} = -\frac{c}{m}dt$

$$\Longrightarrow \int \frac{dv}{v} = -\int \frac{c}{m}dt + K \Longrightarrow \ln|v| = -\frac{c}{m}t + K$$

$$\Longrightarrow v = Ke^{-\frac{c}{m}t} = v_H(t)$$

2. Schritt: (Var. d. Konst.) (1) $v = K(t)e^{-\frac{c}{m}t}$

$$\Longrightarrow (2)\; \dot{v} = \dot{K}e^{-\frac{c}{m}t} - \frac{c}{m}Ke^{-\frac{c}{m}t}$$

$$\underset{(1),(2)\text{ in }(*)}{\Longrightarrow} \dot{K}e^{-\frac{c}{m}t} \cancel{- \frac{c}{m}Ke^{-\frac{c}{m}t}} + \cancel{\frac{c}{m}Ke^{-\frac{c}{m}}} = g$$

$$\Longrightarrow \dot{K} = ge^{\frac{c}{m}t} \Longrightarrow K = g\int e^{\frac{c}{m}t}dt$$

$$= \frac{gm}{c}e^{\frac{c}{m}t} = K(t) \quad (3)$$

(3) in (1) $\Longrightarrow v_p = \frac{gm}{c} \Longrightarrow v_{allg} = v_H + v_p$

$$= Ke^{-\frac{c}{m}t} + \frac{gm}{c} \quad (4)$$

$$v(0) = K + \frac{gm}{c} \overset{!}{=} v_0 \Longleftrightarrow K = v_0 - \frac{gm}{c} \quad (5)$$

(5) in (4) $\Longrightarrow v = \left(v_0 - \frac{gm}{c}\right)e^{-\frac{c}{m}t} + \frac{gm}{c}$

$$= v_0 e^{-\frac{c}{m}t} + \frac{gm}{c}\left(1 - e^{-\frac{c}{m}t}\right) \longrightarrow \frac{gm}{c} \quad (t \to \infty)$$ (Grenzgeschwindigkeit)

Bsp.: <u>(Newton'sches Abkühlungsgesetz)</u>

$$(*) \quad \boxed{\frac{dT}{dt} = -k\,(T - T^*)} \; , \quad \boxed{T(0) = T_0} \qquad (\underline{\text{AWP}})$$

$$\frac{dT}{dt} = -kT + kT^*$$

<u>1. Schritt</u> <u>(hom. DGL)</u> $\quad \frac{dT}{dt} = -kT \iff \frac{dT}{T} = -k\,dt$

$$\underset{\int}{\Longrightarrow} \int \frac{dT}{T} = -k\int dt + K \Rightarrow \quad T = Ke^{-kt} = T_H(t)$$

<u>1. Schritt</u> <u>(Var. d. Konst.)</u> $\quad (1) \quad T = K(t)\,e^{-kt}$

$$\Rightarrow (2) \quad \dot{T} = \dot{K}e^{-kt} - kKe^{-kt}$$

$$\underset{(1),(2)\text{ in }(*)}{\Longrightarrow} \quad \dot{K}e^{-kt} - kKe^{-kt} = -k\cancel{Ke^{-kt}} + kT^*$$

$$\Rightarrow \quad \dot{K} = kT^*e^{kt} \Longrightarrow \quad K = kT^*\int e^{kt}\,dt$$

$$= \frac{kT^*}{k}e^{kt} = T^*e^{kt} \quad (3)$$

$$(3) \text{ in } (1) \Rightarrow \quad T_p = T^*$$

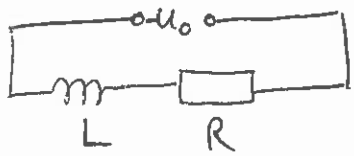
$$\Rightarrow \quad T_{allg} = T_h + T_p = Ke^{-kt} + T^* \quad (4)$$

$$T(0) = K + T^* \overset{!}{=} T_0 \iff \quad K = T_0 - T^* \quad (5)$$

$$\underset{(5)\text{ in }(4)}{\Rightarrow} \quad T = (T_0 - T^*)e^{-kt} + T^* = T_0e^{-kt} + T^*(1 - e^{-kt})$$

$$\longrightarrow T^* \quad (t \to \infty)$$

Bsp.: (RL – Schwingkreis)



(Kirchhoff'sches Gesetz)

$$u_L + u_R = u_0$$

$$\Rightarrow (*) \boxed{L \frac{di}{dt} + Ri = u_0}, \quad \boxed{i(0) = i_0} \quad \text{(AWP)}$$

1\. Schritt: (hom. DGL) $\quad L\frac{di}{dt} + Ri = 0 \Longleftrightarrow \frac{di}{i} = -\frac{R}{L}\,dt$

$$\Rightarrow \int \frac{di}{i} = -\frac{R}{L}\int dt + K \Rightarrow \ln|i| = -\frac{R}{L}t + K$$

$$\Rightarrow i = Ke^{-\frac{R}{L}t} = V_H(t)$$

2\. Schritt (Var. d. Konst.) $\quad$ Gl $\quad i = K(t)e^{-\frac{R}{L}t}$

$$(2) \quad \dot{i} = \dot{K}e^{-\frac{R}{L}t} - \frac{R}{L}Ke^{-\frac{R}{L}t}$$

$$\underset{\text{Gl, (2) in (*)}}{\Longrightarrow} \quad L\cdot\dot{K}e^{-\frac{R}{L}t} \cancel{- L\cdot\frac{R}{L}Ke^{-\frac{R}{L}t}} + \cancel{RKe^{-\frac{R}{L}t}} = u_0$$

$$\Rightarrow \quad \dot{K} = \frac{u_0}{L}\cdot e^{\frac{R}{L}t} \Rightarrow K = \frac{u_0}{L}\int e^{\frac{R}{L}t}\,dt$$

$$= \frac{u_0}{L}\cdot\frac{L}{R}e^{\frac{R}{L}t} = \frac{u_0}{R}e^{\frac{R}{L}t} \quad (3)$$

$$(3) \text{ in } (1) \Rightarrow i_p = \frac{u_0}{R} \Rightarrow i_{allg} = i_H + i_p$$

$$= Ke^{-\frac{R}{L}t} + \frac{u_0}{R}. \quad (4)$$

$$i(0) = K + \frac{u_0}{R} \overset{!}{=} V_0 \Longleftrightarrow K = V_0 - \frac{u_0}{R} \quad (5)$$

$$(5) \text{ in } (4) \Rightarrow \left(i_0 - \frac{u_0}{R}\right)e^{-\frac{R}{L}t} + \frac{u_0}{R} = i(t)$$

$$= i_0 e^{-\frac{R}{L}t} + \frac{u_0}{R}\left(1 - e^{-\frac{R}{L}t}\right) \longrightarrow \frac{u_0}{R} \quad (t \to \infty)$$

Beschränktes Wachstum (Bsp. zu Separation d. Var.)

t ... Zeit
$B(t)$ ... Bestand zum Zp t
S ... Schranke ($S>0$, $k>0$)

$$\boxed{\dot{B}(t) = k(S - B(t))}$$

$$\Rightarrow \frac{\dot{B}}{S-B} = k \iff \frac{dB}{S-B} = k\,dt$$

$$\underset{\int}{\Rightarrow} \int \frac{dB}{S-B} = \int k\,dt + C_1 \Rightarrow -\ln|S-B| = kt + C_1$$

$$\Rightarrow \ln(|S-B|) = -kt - C_1 \underset{\exp}{\Rightarrow} |S-B| = e^{-kt} \cdot \underbrace{e^{-C_1}}_{=:C_2}$$

$$\Rightarrow S - B = \underbrace{\pm C_2}_{=:C} e^{-kt}$$

$$\Rightarrow B(t) = S - C e^{-kt} \qquad \begin{cases} C > 0 : & \underline{\text{Wachstum}} \\ C < 0 : & \underline{\text{Zerfall}} \end{cases}$$

$$B(0) = S - C \overset{!}{=} B_0 \iff C = S - B_0$$

$$\Rightarrow \boxed{B(t) = S - (S - B_0) e^{-kt}} \qquad k>0,\ S>0,\ B_0>0$$

$S - B_0 > 0$: Wachstum

$S - B_0 < 0$: Zerfall