Bsp.: <u>Kugelvolumen</u> (Herleitung über Doppelintegral)

$x^2+y^2+z^2=R^2$: Kugel mit Radius R

$$\Rightarrow z = \pm\sqrt{R^2-x^2-y^2}$$

$$\Rightarrow V_K = 2\iint\limits_{(\text{Kreis})}\sqrt{R^2-x^2-y^2}\,dx\,dy = 2\iint\sqrt{R^2-x^2-y^2}\,dx\,dy =$$

$$= 2\iint\limits_{\substack{-R\le x\le R\\ -\sqrt{R^2-x^2}\le y\le\sqrt{R^2-x^2}}}\sqrt{R^2-x^2-y^2}\,dx\,dy = 2\int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}\,dy\,dx =$$

$$= 2\cdot 2\cdot 2\int_{x=0}^{R}\int_{y=0}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}\,dy\,dx =$$

$$= 8\iint\limits_{(\text{Viertelkreis})}\sqrt{R^2-x^2-y^2}\,dx\,dy \underset{\text{Polarkoord.}}{=} 8\int_{\varphi=0}^{\pi/2}\int_{r=0}^{R}\sqrt{R^2-r^2}\cdot r\,dr\,d\varphi$$

$$\underset{\text{Vert. d. Int.reihenf.}}{=} 8\cdot\frac{\pi}{2}\int_{r=0}^{R}\sqrt{R^2-r^2}\cdot r\,dr = \left|\begin{array}{l} u := R^2-r^2 \\ \frac{du}{dr} = -2r \Rightarrow dr = \frac{du}{-2r} \\ r=0: u=R^2;\quad r=R: u=0\end{array}\right| =$$

$$= 4\pi\int_{u=R^2}^{0}\sqrt{u}\cdot\left(-\frac{1}{2}\right)du = 2\pi\cdot\int_{u=0}^{R^2}u^{1/2}\,du = 2\pi\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^{R^2}$$

$$= \boxed{\tfrac{4}{3}\pi R^3}$$